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Polymer quantization and Symmetries

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Abstract

Polymer quantization was discovered during the construction of Loop Quantum Cosmology. For the simplest quantum theory of one degree of freedom, the implications for dynamics were studied for the harmonic oscillator as well as some other potentials. For more degrees of freedom, the possibility of continuous, kinematic symmetries arises. While these are realised on the Hilbert space of polymer quantum mechanics, their infinitesimal versions are not supported. For an invariant Hamiltonian, these symmetry realizations imply infinite degeneracy suggesting that the symmetry should be spontaneously or explicitly broken. The estimation of symmetry violations in some cases have been analysed before. Here we explore the alternative of shifting the arena to the distributional states. We discuss both the polymer quantum mechanics case as well as polymer quantized scalar field.

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I. INTRODUCTION

In Loop Quantum Gravity [1], the twin demands of SU(2) gauge invariance and diffeomorphism covariance, are met by the use of the holonomies as basic variables and an inner product defined using the Haar measure on SU(2). The Hilbert space one gets is unique up to unitary equivalence [2]. This also has the unusual feature that while the diffeomorphisms have well defined unitary action, their infinitesimal versions cannot be defined as operators [3]. When specialized to the mini-superspace models, analogous procedure leads to the so-called *polymer quantization*[4]. Its Hilbert space is *non-separable* and has the same feature of finite translations being well defined but not the infinitesimal generators - the momenta variables. The Stone-von Neumann Theorem is evaded by relaxing the requirement of (weak) continuity of the representation of the Weyl-Heisenberg algebra. One can however introduce an *approximate* version of momenta and construct corresponding non-relativistic dynamics. This necessarily introduces a fundamental scale and modifies the energy spectra. Nevertheless for certain systems, it can be seen explicitly that the deviations from the usual Schrodinger quantized model are essentially indistinguishable observationally. Such generic conclusions have been obtained for the one dimensional harmonic oscillator [4] and inverse power potentials in (effectively) one dimension [5]. For particles moving in more dimensions, we have the possibilities of rotationally invariant systems and a natural question is to ask how the symmetry can be incorporated. This question, in the more general context of Galilean symmetries has been addressed by Dah-Wei Chiou [6]. He also noted that while finite group actions are well defined, the infinitesimal ones are not. He then explored the 'approximated forms' of the usual generators (which do not form a closed algebra) and concluded that the deviations are small within the domain of validity of the non-relativistic model.

We would like to ask if this technical feature of non-existence of infinitesimal generators has any physically relevant consequences? Is this necessarily an undesirable feature? If yes, how is the role of the polymer representation to be understood? After all in the LQG context, the analogue of polymer representation is very much physically well motivated but forms only an intermediate step due to the constrained nature of the system.

The short answers are that the non-existence of infinitesimal generators of rotations in the polymer representation *also implies infinite degeneracy for any rotationally invariant*

Hamiltonian and this is a physically undesirable feature. One option then is to break the symmetry either explicitly or spontaneously. Yet another possibility is to note that we could view the polymer Hilbert space as part of a Gelfand-like triple, $\mathrm{Cyl} \subset H_{\mathrm{poly}} \subset \mathrm{Cyl}^*$ as in the case of LQG and define infinitesimal generators on a suitable *subspace* of Cyl^*. The usual Schrodinger quantization can then be recovered, albeit trivially. The corresponding steps in the context of a polymer quantized scalar field reveal further possibilities.

In section II we briefly present the basic definitions of polymer quantization in terms of a triple, as well as specify the action of rotations. We point out how rotationally invariant Hamiltonians can be constructed and show that the spectra of such Hamiltonians are infinitely degenerate.

In section III we show how to define infinitesimal generators on a suitable subspace of the dual Cyl^*. A new inner product can be naturally defined on this subspace which makes these generators self-adjoint and also makes the completion unitarily equivalent to the Schrodinger quantization. Although recovering Schrodinger quantization is hardly the aim, we view this as an illustration of a *multi-step* quantization procedure which could be needed in more complex systems.

With this in view, polymer quantized scalar field is considered in the section IV. Although it shares the features seen in polymer quantum mechanics, there seem to be many more possibilities for a quantum theory admitting infinitesimal symmetries.

In the last section V we give a summary and conclude with a discussion. Although the polymer quantization is naturally adapted to diffeomorphism covariance, we have focused exclusively on *symmetries* (as opposed to gauges). Thus we have fixed background coordinates and metric and permitted only the isometries, specifically rotations in three dimensions. We note that the arguments can be easily generalised to other dimensions and other groups including the Lorentz/Poincare groups and point out some of the issues to be addressed in pursuing the Cyl^* avenue.

II. ROTATIONAL INVARIANCE IN POLYMER QUANTUM MECHANICS

Consider a non-relativistic particle moving in three dimensions. Classically it is described by the configuration space, R^3 coordinatized by $\vec{q} \leftrightarrow q^i, i = 1, 2, 3$. To construct Polymer quantization, choose a countable set, γ, of 3-dimensional vectors $\vec{k}_j$ and define a set Cyl_γ of

linear combinations of functions of $\vec{q}$ of the form: $\mathrm{Cyl}_\gamma := \{\sum_j f_j e^{i\vec{k}_j\cdot\vec{q}}, f_j \in \mathbb{C}\}$. Here the coefficients f_j satisfy certain regularity conditions [4] which do not concern us here. Next, define the set of functions of $\vec{q}$, $\mathrm{Cyl} := \cup_\gamma \mathrm{Cyl}_\gamma$. On this set, define the inner product[1],

$$\langle\psi|\psi'\rangle := \lim_{R\to\infty}\frac{3}{4\pi R^3}\int_0^R q^2 dq \sin\theta d\phi\ \psi^*(\vec{q})\psi'(\vec{q}) \quad \Rightarrow \tag{1}$$

$$\langle\vec{k}|\vec{k}'\rangle := \lim_{R\to\infty}\frac{3}{4\pi R^3}\int_0^R q^2 dq \sin\theta d\phi\ e^{i(\vec{k}'-\vec{k})\cdot\vec{q}} = \delta_{\vec{k},\vec{k}'} \quad \forall\ \vec{k},\ \vec{k}' \in \mathbb{R}^3\ . \tag{2}$$

Clearly, $\{e^{i\vec{k}\cdot\vec{q}}\ /\vec{k} \in \mathbb{R}^3\}$ form an uncountable, orthonormal set and we denote them as the kets $|\vec{k}\rangle$. We denote the completion of Cyl w.r.t. this inner product, as $H_{\mathrm{poly}} := \overline{\mathrm{Cyl}}$. The Hilbert space is non-separable and we also have the natural triple, $\mathrm{Cyl} \subset \mathrm{H}_{\mathrm{poly}} \subset \mathrm{Cyl}^*$, where Cyl^* denote the algebraic dual of Cyl. Notice that, the integration measure is invariant under three dimensional rotations and preserves the orthonormality in the second equation above. This will permit unitary representation of rotation group (eqn. 5 below).

It is clear that $\vec{q}$ *cannot* be represented on the polymer Hilbert space as a multiplicative operator since q^i multiplying a basis element does not produce a countable linear combination of the exponentials. The exponentials of the form, e^{ilq^j} however do form multiplicative (and *unitary*) operators. The derivatives too act invariantly on Cyl and $p_i := -i\hbar\frac{\partial}{\partial q^i}$ are self-adjoint operators representing the momenta. The exponentials are the eigenfunctions of the momenta: $\hat{p}_i|\vec{k}\rangle = \hbar k_i|\vec{k}\rangle$.

That the self-adjoint position operators $\hat{q}^i$ do not exist can be seen more formally as well. Consider a 1-parameter family of unitary operators, defined by $\mathcal{U}(\alpha,\vec{m})|\hat{k}\rangle := |\hat{k}+\alpha\vec{m}\rangle\ \forall\ \vec{k} \in \mathbb{R}^3$. For any vector $\vec{\ell}$, $\langle\vec{\ell}|\mathcal{U}(\alpha,\vec{m})|\vec{\ell}\rangle = \langle\vec{\ell}|\vec{\ell}+\alpha\vec{m}\rangle = \delta_{\alpha,0}$, as implied by the orthonormality. Hence, the family of unitary operators is *not* weakly continuous at $\alpha = 0$. *If* a self-adjoint operator of the form $\vec{m}\cdot\vec{q}$ existed, then we could define a one parameter family of unitary operators $\mathcal{V}(\alpha,\vec{m}) := e^{i\alpha\vec{m}\cdot\vec{q}}$ which *is continuous at* $\alpha = 0$ and precisely matches the $\mathcal{U}(\alpha,\vec{m})$ family, thus reaching a contradiction. Hence, on the polymer Hilbert space, the momenta and exponentials of positions are well defined operators but there are *no* self-adjoint operators representing positions. This feature of polymer quantization has profound implications for implementation of continuous, non-abelian symmetries.

[1] Strictly, it is not necessary to give an explicit expression for the inner product. In fact, analogous expression cannot be given when one wants to realise Lorentz symmetry. It is enough to stipulate the orthonormal set, as in eq.(2).

Recall, that any group of symmetries is represented in a quantum theory by *unitary* operators[2], with the states transforming as $|\psi\rangle \to |\psi_g\rangle := U(g)|\psi\rangle$ and the operators transforming as, $A \to A_g := U(g) A\, U^\dagger(g)$ for each group element $g \in \mathcal{G}$. The specific unitary operators representing specific symmetry operation can be determined by *stipulating how the basic observables transform*. For example, with q^i, p_i being the basic observables in the usual quantization, the unitary operators corresponding to rotations, are determined by:

$$q^i_\Lambda := U(\Lambda) q^i\, U(\Lambda)^\dagger = \Lambda^i{}_j q^j \quad , \quad p^\Lambda_i := U(\Lambda) p_i\, U(\Lambda)^\dagger = \Lambda^j{}_i p_j \quad , \quad \Lambda^i{}_m \Lambda^j{}_n \delta^{mn} = \delta^{ij} \tag{3}$$

For infinitesimal rotations, $\Lambda^i{}_j := \delta^i{}_j + \epsilon^i{}_j$, $U(\mathbb{1} + \epsilon) := \mathbb{1} - \frac{i}{\hbar}\epsilon \cdot \hat{J}$ we get,

$$-\ \frac{i}{\hbar}[\epsilon \cdot \hat{J}, q^i] \ = \ \epsilon^i{}_j q^j \quad , \quad -\ \frac{i}{\hbar}[\epsilon \cdot \hat{J}, p_i] \ = \ \epsilon^j{}_i p_j. \tag{4}$$

With the identifications $\epsilon^i{}_j := \epsilon_k \mathcal{E}^{ki}{}_j$, $\epsilon \cdot \hat{J} := \epsilon_k \hat{J}^k$, we deduce $\hat{J}^k := \mathcal{E}_m{}^{nk} q^m p_n$ as the operators representing the infinitesimal generators.

Alternatively, the operators $U(\Lambda)$ could also be determined by specifying their action on *wavefunctions* - explicit functions on the *configuration space* (say), eg. $\Psi_\Lambda(\vec{q}) := \Psi(\overrightarrow{\Lambda q})$.

For the polymer quantization, the defining stipulations for the action of rotations are:

$$\left(e^{i\vec{k}\cdot\vec{q}}\right)_\Lambda := U(\Lambda)\left(e^{i\vec{k}\cdot\vec{q}}\right) U(\Lambda)^\dagger = \left(e^{ik_i\Lambda^i{}_j q^j}\right) \quad , \quad p^\Lambda_i := U(\Lambda) p_i\, U(\Lambda)^\dagger = \Lambda^j{}_i p_j \tag{5}$$

Noting that $|\hat{k}\rangle$ are eigenstates of $\hat{p}_i$, it follows,

$$\begin{aligned} U^\dagger(\Lambda)\hat{p}_i U(\Lambda)|\vec{k}\rangle \ &= \ (\Lambda^{-1})^j{}_i \hat{p}_j |\vec{k}\rangle \ = \ (\Lambda^{-1})^j{}_i k_j |\vec{k}\rangle \\ \therefore \hat{p}_i \left[U(\Lambda)|\vec{k}\rangle\right] \ &= \ \left[(\Lambda^{-1})^j{}_i k_j\right] \left[U(\Lambda)|\vec{k}\rangle\right] \\ \therefore U(\Lambda)|\vec{k}\rangle \ &= \ |(\Lambda^{-1})^j{}_i k_j\rangle \end{aligned} \tag{6}$$

Evidently, this action of rotation group on the polymer Hilbert space is *reducible*, with the orbit through any $\vec{k}$ being spanned by the orthonormal kets $\{|\vec{k}'\rangle\}$ with $\vec{k}'$ lying on the 2-sphere through $\vec{k}$. The subspace spanned by $\{|\vec{k}\rangle, \vec{k}\cdot\vec{k} = \text{constant}\}$, forms an irreducible representation and is clearly infinite dimensional.

This action of the rotation group, coupled with the fact that the kets $|\vec{k}\rangle$ are ortho*normalised*, implies that $U(\Lambda)$ also *cannot* be weakly continuous. Unlike the one dimensional case where the group action necessarily transformed a basis vector to another

[2] We will not be considering time reversal or charge conjugation symmetries, so we will not consider anti-unitary operators.

basis vector, here we have the possibility that $\vec{k}$ could be along the axis of rotation represented by $U(\Lambda)$ and hence invariant under $U(\Lambda)$. To show discontinuity, consider any one parameter subgroup of rotations. All these rotations will leave some particular axis invariant. Choose any $\vec{k}$, orthogonal to this axis. Now the subgroup action transforms a basis ket to another distinct basis ket. The lack of weak continuity for every 1-parameter subgroup follows as before and we cannot write $U(\Lambda) = \mathbb{1} - \frac{i}{\hbar}\epsilon \cdot \hat{J}$. Note that it is *not* the case that *every* one parameter family of unitary operators is necessarily non-continuous. For continuity to be possible, members of the unitary family must *not* map any *basis vector to another basis vector*.

So, while we do not have representation of infinitesimal action, finite rotations are perfectly well defined. However for rotations to be a *symmetry*, their action must also preserve the dynamics. Classically, we have three 'elementary' rotational invariants: $p \cdot p$, $q \cdot q$ and $p \cdot q$. Only first of these can be promoted to operator on the polymer Hilbert space. A Hamiltonian which is a function of p^2 alone will describe only a 'free' dynamics. Is this the only possible rotationally invariant dynamics supported by the polymer Hilbert space? Not quite. As noticed in the context of the 'improved quantization' of LQC, exponentials of arbitrary functions of momenta, times q^i (i.e. functions *linear* in q^i) can also be promoted to well defined operators[3]. This is because the Hamiltonian vector field X_{q^i} generates translations along p_i and any function of $\vec{p}$ multiplying X_{q^i} generates more general infinitesimal transformations, also along p_i. While X_{q^i} cannot be promoted to an operator, its exponential which generates *finite diffeomorphisms* can be! Incorporating rotational invariance, we can thus have unitary operators of the form $e^{\pm i f(p^2)\, p_i q^i}$. From these, the corresponding *sin* and *cos* self-adjoint operators can be defined. A candidate rotationally invariant Hamiltonian will be a function of p^2 and the sin, cos operators. There is no corresponding trick to use the $q \cdot q$ invariant.

To compute the action of finite diffeomorphism, say by unit parameter, consider the integral curves defined by,

$$\frac{dp_i}{d\lambda} \;=\; f(p\cdot p)\, p_i \quad \Rightarrow \quad \frac{dp\cdot p}{d\lambda} = 2(p\cdot p) f(p\cdot p) \tag{7}$$

This defines the change in the $p \cdot p$ for unit change in the parameter. Notice that the vector

[3] We thank Alok Laddha for pointing this out.

field is *radial*, and therefore the integral curves are in the radial direction (in 'p'-space) and for unit change in the parameter, connect two spheres of radii $p^2_{\rm initial}$ and $p^2_{\rm final} := \xi^2 p^2_{\rm initial}$. The corresponding unitary operator is then defined by,

$$\widehat{e^{-if(p^2)p\cdot q}}|\vec{k}\rangle := |\vec{k}' = \xi\vec{k}\rangle \ , \tag{8}$$

the scale ξ being determined by eq.(7).

Thus, we *can* have non-trivial rotationally invariant dynamics. However, there is now a different problem. As noted before, the unitary representation of SO(3) on the polymer Hilbert space is reducible with irreducible representations carried by $\mathcal{H}_\sigma := \text{span}\{|\vec{k}\rangle\, , k\cdot k = \sigma^2 > 0\}$. Each of these is infinite dimensional. Each eigenspace of any invariant Hamiltonian will carry a representation of SO(3) which has to be infinite dimensional, being made up of some of the irreducible representations together possibly with the trivial representation ($\sigma = 0$). Thus we face the problem of *infinite degeneracy* which is physically untenable: the partition function of such a system will be undefined. We have now two possibilities: (a) rotations cease to be a symmetry (explicit breaking of symmetry) or (b) *spontaneous breaking* of rotational symmetry.

To see both possibilities, we first seek an approximate substitute for the position operators. The operators, $e^{i\vec{k}\cdot\vec{q}}$, allow us to define families of self-adjoint operators. For instance, choosing $\vec{k}_j := \delta\hat{e}_j,\ \hat{e}_j$ a unit vector, we can define $\sin_{\delta\hat{\rm e}_{\rm j}} := (2{\rm i})^{-1}({\rm e}^{{\rm i}\delta\hat{\rm e}_{\rm j}\cdot\vec{\rm q}} - {\rm e}^{-{\rm i}\delta\hat{\rm e}_{\rm j}\cdot\vec{\rm q}})$ and a cos operator analogously[4]. We could choose several triplets of linearly independent unit vectors $\hat{e}_j$ and also choose many different parameters δ's (equivalently, finitely many $\vec{k}_j$). If we collect finitely many of such sets and restrict ourselves to observables which are functions of these (and the momentum) operators, then from any given $|\vec{k}_0\rangle$, we will generate a collection of basis vectors, $\{|\vec{k}_0 + \sum_j n_j\vec{k}_j\rangle, n_j \in {\rm Z}\!\!{\rm Z}\}$. The closed subspace generated by this set will be a *proper subspace* of the polymer Hilbert space and is clearly *separable*. If we also include operators which are exponentials in $p\cdot q$, discussed above, then the lattice generated will also involve scaling determined by the choices for $f(p^2)$. As long as the number of such operators is finite, we will continue to have separable sectors. The chosen set of observables, will act invariantly on each of these subspaces and will provide *superselection sectors*. Observe that among the chosen class of observables, we can also have an invariant Hamiltonian. Action

[4] These operators however do not suffice to represent the Lie algebra of rotations [6].

of rotations however mixes different sectors and we have *spontaneous breaking of rotational invariance*. If we chose a Hamiltonian involving the approximated position operators, we have explicit breaking of rotations controlled by the δ-parameter(s). The example of spherically symmetric harmonic oscillator in three dimensions illustrates this. For a economical parametrization of violation, we can choose a single common δ. For sufficiently small values of this, at a certain level of observational precision, it is of course possible to have the illusion of rotational invariance.

To summarise, having made a choice of the polymer Hilbert space $H_{\rm poly}$, we *can* have exact rotational symmetry with a some what restricted form of dynamics (no $q \cdot q$ dependence) but with uncountably infinite degeneracy. To avoid the problem of infinite degeneracy, the symmetry must be broken - either explicitly *or* spontaneously. By introducing separable sectors we can see both possibilities.

One can however also view the polymer quantization as an intermediate step in constructing a quantum theory, much as the kinematical Hilbert space of LQG is. Using the triple, we can try to select a suitable *subspace* of Cyl* on which infinitesimal generators can be defined. With a suitable choice of a new inner product, we can obtain a 'physical' Hilbert space with a rotationally invariant dynamics.

III. INFINITESIMAL GENERATORS

The possibility of looking to Cyl* for a home to a suitable quantum theory is inspired by analogous steps taken in the context of LQG. In LQG, the step is motivated for a very different reason. The kinematical quantization is essentially forced upon us by the demand of SU(2) invariance and diffeomorphism covariance. Since there are constraints whose kernels are in general distributional, an appropriate diffeo-invariant subspace of corresponding Cyl* is a natural arena. In our case, the polymer quantization itself is not a compulsion, but is a useful illustration of a *multi-step construction of a quantum theory.*

Recall that construction of $\mathrm{H}_{\rm poly}$ naturally gave us the triple $\mathrm{Cyl} \subset \mathrm{H}_{\rm poly} \subset \mathrm{Cyl}^*$. This structure provides us with a convenient representation of the elements $(\Psi|$ of Cyl* by complex valued linear functions $\psi^*(\vec{\mathrm{k}}) := (\Psi|\vec{\mathrm{k}}\rangle$. No smoothness properties are assumed at this stage for these functions. Furthermore, for every operator $\mathrm{A} : \mathrm{Cyl} \to \mathrm{Cyl}$, we can define an operator $\tilde{\mathrm{A}} : \mathrm{Cyl}^* \to \mathrm{Cyl}^*$ by the 'dual action', eg. $(\tilde{\mathrm{A}}\Psi|\mathrm{f}\rangle := (\Psi|\mathrm{Af}\rangle, \forall\ |\mathrm{f}\rangle \in \mathrm{Cyl},\ \forall\ (\Psi| \in \mathrm{Cyl}^*$.

Conversely, given an operator $\tilde{\mathrm{A}}$ defined on *all* of Cyl*, we can define an operator A on Cyl by the same equation as above (read backwards). In particular this means that we have the operators $\tilde{\mathrm{U}}(\Lambda)$ defined on Cyl*. We will use these to define infinitesimal generators on Cyl*. We will also define the position operators.

We begin with infinitesimal rotation generators.

$$\begin{aligned}
(\Psi|\mathrm{U}(1+\epsilon)-\mathrm{U}(1-\epsilon)|\vec{\mathrm{k}}\rangle &= (\Psi|\vec{\mathrm{k}}+\vec{\epsilon\mathrm{k}}\rangle-(\Psi|\vec{k}-\vec{\epsilon\mathrm{k}}\rangle \\
&\approx 2\epsilon_{\mathrm{l}}\mathcal{E}^{\mathrm{li}}{}_{\mathrm{j}}\mathrm{k}^{\mathrm{j}}\frac{\partial\psi^*}{\partial\mathrm{k}^{\mathrm{i}}} \\
\therefore \lim_{\epsilon_{\mathrm{l}}\to 0}(\Psi|\frac{\mathrm{U}(1+\epsilon)-\mathrm{U}(1-\epsilon)}{2\epsilon_{\mathrm{l}}}|\vec{\mathrm{k}}\rangle &= \mathcal{E}^{\mathrm{li}}{}_{\mathrm{j}}\mathrm{k}^{\mathrm{j}}\frac{\partial\psi^*}{\partial\mathrm{k}^{\mathrm{i}}} \\
\therefore (\mathrm{J}^{\mathrm{l}}\Psi|\vec{\mathrm{k}}\rangle &:= -\mathrm{i}\hbar\mathcal{E}^{\mathrm{li}}{}_{\mathrm{j}}\ \mathrm{k}^{\mathrm{j}}\ \frac{\partial\psi^*}{\partial\mathrm{k}^{\mathrm{i}}} \qquad (9)
\end{aligned}$$

Notice that these operators are defined on a *subspace* of Cyl*, consisting of those $(\Psi|$ whose corresponding $\psi^*(\vec{\mathrm{k}})$ are differentiable functions. Hence, by dual action we *cannot* define the corresponding operators on Cyl.

Next, recall the $\sin_{\delta\hat{\mathrm{e}}_{\mathrm{j}}}$ operators defined in the previous section. For each orthonormal triad, $\hat{e}_j,\ j=1,2,3,\ \hat{e}_i\cdot\hat{e}_j=\delta_{ij}$ and a small parameter δ, we have, $U_{\delta\hat{e}_j}(\vec{q}) := e^{i\delta\hat{e}_j\cdot\vec{q}}$ and $\sin_{\delta\hat{e}_j} := (2i)^{-1}(U_{\delta\hat{e}_j}(\vec{q})-U_{-\delta\hat{e}_j}(\vec{q}))$. Now,

$$\begin{aligned}
2\mathrm{i}(\Psi|\sin_{\delta\hat{\mathrm{e}}_{\mathrm{j}}}|\vec{\mathrm{k}}\rangle &= (\Psi|\vec{\mathrm{k}}+\delta\hat{\mathrm{e}}_{\mathrm{j}}\rangle-(\Psi|\vec{\mathrm{k}}-\delta\hat{\mathrm{e}}_{\mathrm{j}}\rangle \\
&= \psi^*(\vec{\mathrm{k}}+\delta\hat{\mathrm{e}}_{\mathrm{j}})-\psi^*(\vec{\mathrm{k}}-\delta\hat{\mathrm{e}}_{\mathrm{j}}) \\
&\approx 2\delta\hat{\mathrm{e}}_{\mathrm{j}}\frac{\partial\psi^*}{\partial\vec{\mathrm{k}}} \\
\therefore \lim_{\delta\to 0}(\Psi|\frac{\sin_{\delta\hat{\mathrm{e}}_{\mathrm{j}}}}{\delta}|\vec{\mathrm{k}}\rangle &= -\mathrm{i}\hat{\mathrm{e}}_{\mathrm{j}}\cdot\vec{\nabla}_{\vec{\mathrm{k}}}\psi^* \qquad (10)
\end{aligned}$$

By restricting to functions ψ^* to those which are differentiable, we can define a *position operator* on a *subspace* of Cyl* via the dual action,

$$(\hat{\mathrm{e}}_{\mathrm{j}}\cdot\vec{\mathrm{q}}\ \Psi|\vec{\mathrm{k}}\rangle := -\mathrm{i}\hat{\mathrm{e}}_{\mathrm{j}}\cdot\vec{\nabla}_{\vec{\mathrm{k}}}\psi^*\ ,\ \forall\ (\Psi|\in\mathrm{Cyl}^*\ \text{ such that } \psi^*(\vec{\mathrm{k}})\text{ is differentiable .} \qquad (11)$$

It is easy to see that the position operators defined above and the momentum operators defined by dual action, also satisfy,

$$(\ [\hat{\mathrm{e}}_{\mathrm{m}}\cdot\vec{\mathrm{q}}\ ,\ \hat{\mathrm{e}}_{\mathrm{n}}\cdot\vec{\mathrm{p}}]\ \Psi|\vec{\mathrm{k}}\rangle = (\ \{\mathrm{i}\hbar\hat{\mathrm{e}}_{\mathrm{m}}\cdot\hat{\mathrm{e}}_{\mathrm{n}}\}\ \Psi|\vec{\mathrm{k}}\rangle\ .$$

The usual strategy for choosing an inner product on Cyl* is to require that a specified set of operators are self-adjoint. In the present case, the choice of operators is obvious - $\hat{e}_m\cdot\vec{q}$ and

$\hat{e}_n \cdot \vec{p}$ - for an orthonormal triad $\hat{e}_j$. If such an inner product exists, then it follows immediately that the above commutation relation will lead to a weakly continuous representation of the usual Weyl-Heisenberg algebra. This representation is unique up to unitary equivalence and can be constructed by completing a Schwartz space of functions $\psi^*(\vec{k})$ with the usual square integrability with respect to the Lebesgue measure, d^3k. Thus the subspace of Cyl* that we can choose is naturally suggested along with the inner product! Of course we have just constructed back the usual Schrodinger quantization using functions of "momenta", $\vec{k}$ instead of functions of "positions", $\vec{q}$. The intermediate polymer quantization has only led us to the Heisenberg representation instead of the Schrodinger representation.

This is obviously a roundabout way of arriving at the usual quantization. But it shows that (a) not every choice of quantization may be flexible enough for physical modeling and (b) we can reach a satisfactory quantum theory by modifying the quantization algorithm. In principle, if the quantum theory constructed from a subspace of Cyl* were *not* satisfactory, we could repeat the process forming a new triple. This is further discussed in the last section. In the next section we discuss the case of a scalar field theory.

IV. THE CASE OF A SCALAR FIELD THEORY

Can rotational invariance be supported in a 'polymerised scalar field theory'? Consider the example of a scalar field $\phi(\vec{x})$ defined on R^3. The rotations act on the space which in turn induces a transformation on the field: $\phi'(\vec{x}) = \phi(\overrightarrow{\Lambda x})$. The polymer quantization of the scalar field is done as follows [7].

Define a vertex set $V = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n)$, of finitely many, *distinct* points. For *non-zero* real numbers $\lambda_j, j = 1, \ldots, n$, define the functions $\mathcal{N}_{V,\vec{\lambda}}(\phi) := e^{i\sum_j \lambda_j \phi(\vec{x}_j)}$. For each fixed set V, let Cyl$_V$ denote the set of finite, complex linear combinations of these functions. Let Cyl $:= \cup_V$ Cyl$_V$. Thus every element of Cyl is a function of ϕ which is a *finite* linear combination of functions $\mathcal{N}_{V,\vec{\lambda}}$ for *some* vertex set V and *some* choice of $\vec{\lambda}$. Define an inner product,

$$\langle\psi|\psi'\rangle := \int d\mu(\phi)\psi^*(\phi)\psi'(\phi) = \int d\mu(\phi) \sum_{V,\vec{\lambda},V',\vec{\lambda}'} C^*_{V,\vec{\lambda}} C'_{V',\vec{\lambda}'} e^{i\sum_k \lambda'_k \phi(\vec{x}'_k) - i\sum_j \lambda_j \phi(\vec{x}_j)} \tag{12}$$

$$= \sum_{V,\vec{\lambda},V',\vec{\lambda}'} C^*_{V,\vec{\lambda}}\, C_{V',\vec{\lambda}'} \quad , \quad \text{the sum is over those terms whose exponent vanishes } \forall\ \phi.$$

The exponent in a term vanishes identically only if the λ corresponding to a $\phi(\vec{x})$ vanishes.

Observe that each term in the summand is again of the form $\mathcal{N}_{V\cup V',\vec{\lambda},\vec{\lambda}'}$, except that all vertices in the union $V\cup V'$ are not necessarily distinct. If $\vec{x}'_k = \vec{x}_j$, then the exponent would be $(\lambda'_k - \lambda_j)\phi(\vec{x}_j)$. If the $\lambda' s$ are equal, then the exponent is identically zero and the integral contributes to the sum. Otherwise, the integral gives zero. It follows that $\mathcal{N}_{V,\vec{\lambda}}$ and $\mathcal{N}_{V',\vec{\lambda}'}$ are orthogonal unless the two sets of vertices coincide *and* their corresponding λ's are equal.

The Hilbert space $\mathrm{H}_{\mathrm{poly}}$, is obtained as the Cauchy completion of Cyl with respect to this inner product. The functions $\mathcal{N}_{V,\vec{\lambda}}(\phi)$, with *every* $\lambda \neq 0$, form an orthonormal basis for the polymer Hilbert space. The constant function corresponding to empty vertex set, $\mathcal{N}(\phi) = 1$, is also included in the basis.

Action of rotations on Cyl is defined by $[U_\Lambda \psi](\phi) := \psi(\Lambda \circ \phi)$. Evaluating it on the elementary functions lead to,

$$\mathcal{N}_{V,\vec{\lambda}}(\phi) \;\rightarrow\; \mathcal{N}'_{V',\vec{\lambda}'}(\phi) \;:=\; \mathcal{N}_{V,\vec{\lambda}}(\phi') \;=\; \mathcal{N}_{V',\vec{\lambda}}(\phi) \tag{13}$$

The middle equality is the definition of the action, $\phi' = \Lambda \circ \phi$ and we have used the scalar nature of ϕ, $\phi'(\vec{x}) = \phi(\overrightarrow{\Lambda x})$, in the last equality.

Observe that under the action of rotation Λ, a vertex set $V = (\vec{x}_1, \ldots, \vec{x}_n)$ changes to a new vertex set $V' := (\overrightarrow{\Lambda x_1}, \ldots, \overrightarrow{\Lambda x_n})$. The $\lambda' s$ are unchanged and the field is evaluated at the transformed points. Since the λ's do not change and the inner product depends only on them, the inner product among elementary functions is invariant under the action of the rotations and therefore rotations are represented unitarily on the Hilbert space.

That this unitary action is also non-weakly-continuous can be seen easily. For non-trivial rotation, a diagonal matrix element between basis states is zero while for the identity rotation, the matrix element is 1. Thus, infinitesimal generators have no representation on the polymer Hilbert space.

The momenta variables are defined as,

$$P_g := \int d^3x g(\vec{x})\pi_\phi(x) \;=\; -i\hbar \int d^3x g(\vec{x}) \frac{\delta}{\delta\phi(\vec{x})} \tag{14}$$

Here $g(\vec{x})$ is a 'suitably smooth' function (π_ϕ has density weight 1, though not relevant here). It is easy to see that,

$$P_g \mathcal{N}_{V,\vec{\lambda}} \;=\; \left[\hbar \sum_j \lambda_j g(\vec{x}_j)\right] \mathcal{N}_{V,\vec{\lambda}} \;\;,\;\; [P_f, P_g] = 0 \;\;,\;\; P_f^\dagger \;=\; P_f \;. \tag{15}$$

Thus the momentum representation exists and the elementary functions $\mathcal{N}_{V,\vec{\lambda}}$ are simultaneous eigenstates of the momenta variables P_g. Under the action of rotation, $U(\Lambda)$, the momentum variables transform as,

$$\begin{aligned} U_\Lambda P_g(\pi) U_\Lambda^\dagger &:= P_g(\Lambda \circ \pi) \\ &= P_{\Lambda^{-1}\circ g}(\pi) \quad \text{(from the definition)} \quad \Rightarrow \\ U_\Lambda P_g(\pi) &= P_{\Lambda^{-1}\circ g}(\pi) U_\Lambda \end{aligned} \tag{16}$$

This is consistent with (13). Let us use the notation $|V,\vec{\lambda}\rangle \leftrightarrow \mathcal{N}_{V,\vec{\lambda}}(\phi)$.

Observe that $e^{i\lambda\phi(\vec{x})}$, a '*point holonomy operator*', clearly acts as a multiplication operator:

$$e^{i\lambda\phi(\vec{x})}|V,\vec{\lambda}\rangle := \begin{cases} |\vec{x}_1,\ldots,\vec{x}_n,\vec{x}\,;\lambda_1,\ldots,\lambda_n,\lambda\rangle & \text{if } \vec{x}\neq\vec{x}_i \text{ for any i} \\ |\vec{x}_1,\ldots,\vec{x}_k\ldots\vec{x}_n\,;\lambda_1,\ldots,\lambda_k+\lambda,\ldots\lambda_n\rangle & \text{if } \vec{x}=\vec{x}_k\ ,\ \lambda+\lambda_k\neq 0 \\ |\vec{x}_1,\ldots,,\ldots\vec{x}_n\,;\lambda_1,\ldots,,\ldots\lambda_n\rangle & \text{if } \vec{x}=\vec{x}_k\ ,\ \lambda+\lambda_k=0 \end{cases} \tag{17}$$

In the last equation, the $\vec{x}_k,\lambda_k$ labels are missing on the right hand side.

What about the scalar field operator itself? It does *not* exist since the point holonomy operators are not weakly continuous, exactly as in the point particle case. Smearing it with a suitable function also does not help to give an operator on the Hilbert space.

Now consider an element $(\Psi| \in \mathrm{Cyl}^*$. Its action on an elementary function $\mathcal{N}_{V,\vec{\lambda}}(\phi)$ is given by,

$$(\Psi|V,\vec{\lambda}\rangle =: \psi^*(\vec{x}_1,\cdots,\vec{x}_n,\lambda_1,\cdots,\lambda_n) \quad , \quad \text{distinct } \vec{x}\,'s \text{ and non-zero } \lambda' s.$$

Under the action of rotations, the arguments of the elementary function change: $|V,\vec{\lambda}\rangle \to |V',\vec{\lambda}\rangle$. Thus, if we choose the functions ψ^*'s to be differentiable, we can define infinitesimal rotations as operators on a subspace of Cyl^*, exactly as before. Explicitly,

$$\begin{aligned} (\Psi|\mathrm{U}(1+\epsilon)-\mathrm{U}(1-\epsilon)|\mathrm{V},\vec{\lambda}\rangle &= (\Psi|\mathrm{V}'_+,\vec{\lambda}\rangle - (\Psi|\mathrm{V}'_-,\vec{\lambda}\rangle \\ &= \psi^*(\vec{\mathrm{x}}_1+\overrightarrow{\epsilon x_1},\cdots,\vec{\mathrm{x}}_\mathrm{n}+\overrightarrow{\epsilon x_n},\vec{\lambda}) - \\ &\qquad \psi^*(\vec{\mathrm{x}}_1-\overrightarrow{\epsilon x_1},\cdots,\vec{\mathrm{x}}_\mathrm{n}-\overrightarrow{\epsilon x_n},\vec{\lambda}) \\ &\approx 2\epsilon_\mathrm{k}\mathcal{E}^{\mathrm{ki}}{}_\mathrm{j}\sum_{\mathrm{m}=1}^{\mathrm{n}}\mathrm{x}^\mathrm{j}_\mathrm{m}\frac{\partial\psi^*}{\partial\mathrm{x}^\mathrm{i}_\mathrm{m}} \\ \therefore \lim_{\epsilon_\mathrm{k}\to 0}(\Psi|\frac{\mathrm{U}(1+\epsilon)-\mathrm{U}(1-\epsilon)}{2\epsilon_\mathrm{k}}|\mathrm{V},\vec{\lambda}\rangle &= \mathcal{E}^{\mathrm{ki}}{}_\mathrm{j}\sum_{\mathrm{m}=1}^{\mathrm{n}}\mathrm{x}^\mathrm{j}_\mathrm{m}\frac{\partial\psi^*}{\partial\mathrm{x}^\mathrm{i}_\mathrm{m}} \\ \therefore (J^k\Psi|V,\vec{\lambda}\rangle &:= -\mathrm{i}\hbar\,\mathcal{E}^{\mathrm{ki}}{}_\mathrm{j}\sum_{\mathrm{m}=1}^{\mathrm{n}}\mathrm{x}^\mathrm{j}_\mathrm{m}\frac{\partial\psi^*}{\partial\mathrm{x}^\mathrm{i}_\mathrm{m}} \end{aligned} \tag{18}$$

Thus, by restricting to a subspace of Cyl*, corresponding to suitably differentiable functions $\psi^*(V,\vec{\lambda})$, we can define the generator of the infinitesimal rotations.

Likewise, to define a smeared operator scalar field on Cyl*, consider,

$$\begin{aligned}(\Psi|\phi^\delta_f|V,\vec{\lambda}\rangle \;&:=\; \int d^3x f(\vec{x})(\Psi|\frac{e^{i\delta\phi(\vec{x})}-e^{-i\delta\phi(\vec{x})}}{2i\delta}|V,\vec{\lambda}\rangle \\ &=\; \int d^3x \frac{f(\vec{x})}{2i\delta}\Big((\Psi|V,\vec{x},\vec{\lambda},\delta\rangle - (\Psi|V,\vec{x},\vec{\lambda},-\delta\rangle\Big)\end{aligned} \tag{19}$$

For a generic $\vec{x}$, assuming differentiability of ψ^*, we will get a function of the vertices of V and the corresponding $\lambda' s$ together with the additional point $\vec{x}$ and the corresponding 'δ' $=$ 0. This function cannot come from any element of Cyl* acting on $|V,\vec{\lambda}\rangle$. Hence we should avoid getting a contribution from a generic $\vec{x}$. If however, $\vec{x}$ coincides with one of the vertices in V, then the resultant function (derivative) *is* a function of $(V,\vec{\lambda})$ and we can interpret the right hand side as a new element of Cyl* evaluated on the basis element $|V,\vec{\lambda}\rangle$. This can be made more precise by employing the commonly used procedure of defining the integral by introducing a cell decomposition adapted to the 'graph' (vertices of V) and *demanding that*

$$\frac{\partial\psi^*}{\partial\lambda_j}(\vec{x}_1,\ldots,\vec{x}_n,\lambda_1,\ldots,\lambda_j,\ldots,\lambda_n)\Big|_{\lambda_j=0} = 0 \quad ,\forall\; j=1,2,\ldots,n\;. \tag{20}$$

This condition ensures that there is no contribution from cells that do not contain a vertex of V and we are led to the definition:

$$(\widetilde{\phi_f}\Psi|V,\vec{\lambda}\rangle \;:=\; \lim_{\delta\to 0}(\Psi|\phi^\delta_f|V,\vec{\lambda}\rangle \;:=\; -i\sum_j f(\vec{x}_j)\frac{\partial\psi^*(\vec{x}_1,\ldots,\vec{x}_n,\lambda_1,\ldots,\lambda_n)}{\partial\lambda_j}\;. \tag{21}$$

It is now easy to verify that

$$\begin{aligned}(\widetilde{[\phi_f,P_g]}\Psi|V,\vec{\lambda}\rangle \;&:=\; (\widetilde{P_g}\widetilde{\phi_f}\Psi|V,\vec{\lambda}\rangle - (\widetilde{\phi_f}\widetilde{P_g}\Psi|V,\vec{\lambda}\rangle \\ &=\; -i\hbar\left(\sum_{j=1}^n f(\vec{x}_j)g(\vec{x}_j)\right)\;(\Psi|V,\vec{\lambda}\rangle \\ &=\; (\left\{+i\hbar\left(\sum_{j=1}^n f(\vec{x}_j)g(\vec{x}_j)\right)\right\}\Psi|V,\vec{\lambda}\rangle\;.\end{aligned} \tag{22}$$

We have thus succeeded in defining the smeared versions of the field operators ϕ_f, P_g in a subspace of Cyl*.

We can also verify that the infinitesimal generators J^k induce expected actions on the smeared fields operators.

$$(\widetilde{[J^k,\phi_f]}\Psi|V,\vec{\lambda}\rangle \;=\; (\widetilde{\phi_f}\widetilde{J_k}\Psi|V,\vec{\lambda}\rangle - (\widetilde{J_k}\widetilde{\phi_f}\Psi|V,\vec{\lambda}\rangle$$

$$
\begin{aligned}
&= -i\sum_{m=1}^{N} f(\vec{x}_m)\frac{\partial\psi^*_{J_k}(V,\vec{\lambda})}{\partial\lambda_m} - (-i\hbar)\mathcal{E}^{ki}{}_j\sum_{n=1}^{N} x^j_n\frac{\partial\psi^*_{\phi_f}(V,\vec{\lambda})}{\partial x^i_n}\\
&= -\hbar\sum_{m,n=1}^{N} f(\vec{x}_m)\mathcal{E}^{ki}{}_j x^j_n\frac{\partial\psi^*(V,\vec{\lambda})}{\partial x^i_n}\\
&\qquad\qquad + \ \hbar\mathcal{E}^{ki}{}_j\sum_{m,n=1}^{N} x^j_n\frac{\partial}{\partial x^i_n}\left\{f(\vec{x}_m)\frac{\partial\psi^*(V,\vec{\lambda})}{\partial\lambda_m}\right\}\\
&= i\hbar\left[-i\sum_{n=1}^{N}\left(\mathcal{E}^{ki}{}_j x^j_n\frac{\partial f(\vec{x}_n)}{\partial x^i_n}\right)\cdot\frac{\partial\psi^*(V,\vec{\lambda})}{\partial\lambda_n}\right]\\
&= i\hbar\ (\widetilde{\phi_{\mathcal{L}_k f}}\Psi|V,\vec{\lambda}\rangle \qquad , \qquad \mathcal{L}_k f(\vec{x}) := \mathcal{E}^{ki}{}_j x^j\frac{\partial f}{\partial x^i} \qquad (23)
\end{aligned}
$$

Similar computation can be done for commutator of $[J^k, P_g]$.

We have now identified the *minimal conditions*, namely differentiability in all arguments and the condition of equation (20), on functions $\psi^*(\vec{x}_1,\ldots,\vec{x}_n,\lambda_1,\ldots,\lambda_n)$ in order that the smeared field operators and the infinitesimal rotation actions are well defined. Since such an element of Cyl* can be viewed as a sequence of *differentiable*, complex functions defined on $(\mathrm{R}^{3n} - \text{diagonal})\times(\mathrm{R}^n - \vec{0})$ where diagonal is the subset of R^{3n} with two or more points coinciding, we are restricted to a subspace of Cyl*. The next step is to choose a suitable inner product on this subspace, possibly restricted further with additional conditions. Let us denote such a subspace by Cyl$_1$. Here we initiate first steps. For notational simplicity, let us denote elements of Cyl* generically by underlined letters such as as $\underline{\Psi}, \underline{\Phi}, \underline{[V,\vec{\lambda}]}, \ldots$ etc.

Heuristically, we can represent each element of Cyl$_1$ and a yet to be defined inner product as,

$$
\begin{aligned}
\underline{\Psi} &:= \sum_{V,\vec{\lambda}}\psi^*(V,\vec{\lambda})\underline{[V,\vec{\lambda}]}\ , \qquad (24)\\
\langle\underline{\Psi},\underline{\Phi}\rangle &:= \sum_{V,\vec{\lambda}}\sum_{V',\vec{\lambda}'}\psi(V,\vec{\lambda})\phi^*(V',\vec{\lambda}')\langle\underline{[V,\vec{\lambda}]},\underline{[V',\vec{\lambda}']}\rangle\\
&:= \sum_{V,\vec{\lambda}}\sum_{V',\vec{\lambda}'}\psi(V,\vec{\lambda})\phi^*(V',\vec{\lambda}')\mathcal{G}(V,\vec{\lambda};V',\vec{\lambda}') \qquad (25)
\end{aligned}
$$

The coefficients $\psi^*(V,\vec{\lambda})$ in the first line, contain the information about the subspace, Cyl$_1$. The $\mathcal{G}$ denotes the inner product between 'basis' elements.

For example, Cyl is a subspace of Cyl* through the natural embedding $|V,\vec{\lambda}\rangle \in$ Cyl $\to \underline{[V,\vec{\lambda}]} \in$ Cyl*. If Cyl$_1$ were to be this subspace, then the $\psi^*(V,\vec{\lambda})$ in eqn.(24) would be non-zero only for finitely many $(V,\vec{\lambda})$ sets and $\mathcal{G}(V,\vec{\lambda};V',\vec{\lambda}')$ would equal $\delta_{V,V'}\delta_{\vec{\lambda},\vec{\lambda}'}$. The double

summation would then collapse to a *finite* sum over $(V, \vec{\lambda})$ (compare eqn.(12)). Likewise, if Cyl_1 were to echo the Hilbert space of the r-Fock construction [7, 8], the $\mathcal{G}(V, \vec{\lambda}; V', \vec{\lambda}')$ would be $\sim \exp[-\frac{1}{4}\sum_{ij} G_{ij}(\vec{x}_i, \vec{x}_j)\lambda^i\lambda^j]$, where the sum over (i, j) is over the vertices of $V \cup V'$ and we use the notation of [7]. The double sum will be a finite sum since $\psi^*(V, \vec{\lambda})$ is non-zero for finitely many $(V, \vec{\lambda})$ sets.

More generally, we could have uncountably many non-zero $\psi^*(V, \vec{\lambda})$ and then each $\underline{\Psi}$ can be thought of as a potentially infinite sequence of functions, ψ_n, on $\sim \mathrm{R}^{4n}$. If we choose an inner product so that the 'basis states' are orthonormal ($\mathcal{G} \propto \delta_{V,V'}\delta_{\vec{\lambda},\vec{\lambda}'}$), then we may write the inner product as,

$$\begin{aligned} \langle \underline{\Psi}, \underline{\Phi} \rangle &:= \sum_{V,\vec{\lambda}} \psi(V, \vec{\lambda})\phi^*(V', \vec{\lambda}') \qquad (26) \\ &:\simeq \sum_{n=0}^{\infty} \int_{\mathrm{R}^{3n}} d^{3n}x \int_{\mathrm{R}^n} d^n\lambda \; \psi(\vec{x}_1, \ldots, \vec{x}_n, \lambda_1, \ldots, \lambda_n)\phi^*(\vec{x}_1, \ldots, \vec{x}_n, \lambda_1, \ldots, \lambda_n) \end{aligned}$$

The $:\sim$ indicates that the integration measures need to be defined and we need to put conditions to ensure convergence of the sum (27).

Assuming that we can choose suitable weights in the sum and measures in the integrations, what further conditions we need to put on the ψ_n's so that our basic operators and generators are self-adjoint? It is easy to see that we need only the usual fall-off conditions on these so that surface terms resulting from the partial integrations drop out. Roughly, we make each member ψ_n as an element of $L_2(\mathrm{R}^{4n})$. This indicates that it is, at least heuristically, possible to choose suitable definitions to construct a new Hilbert space. A more detailed analysis will be considered elsewhere.

V. SUMMARY AND DISCUSSION

We began by exploring symmetries and their violations in polymer quantized systems. Specifically, we focused on three dimensional rotations and explored the polymer quantized particle in three dimensions and a scalar field defined on R^3. It is certainly possible to have a unitary representation of SO(3) on the polymer Hilbert space but the representation is discontinuous and consequently does not admit representation of its Lie algebra. The non-availability of configuration space operators - position operators - severely restricts the possible invariant Hamiltonians and *every one of these* has infinitely degenerate eigenvalues.

In effect, physically acceptable dynamics on polymer Hilbert space *must necessarily violate rotational symmetry, either explicitly or spontaneously.* In case of explicit breaking, one can then look for economical parametrization of symmetry violations and put bounds on the parameters. As noted in the introduction, this route has already been followed in [4–6]. We explored another route to see if acceptable quantization, *with infinitesimal symmetries*, can be arrived at viewing polymer quantization as an intermediate step. This was done by looking for suitable subspace(s) of the dual member of the Gelfand triple with a hope of defining a new inner product and a new Hilbert space. For the point particle case we verified that it is possible to construct a new Hilbert space which carries continuous representations of the rotation group as well as continuous representation of the Heisenberg group. By the Stone-von Neumann theorem, this is of course the usual Schrodinger representation which supports the usual rotationally invariant non-trivial Hamiltonians. The case of scalar field revealed greater richness. There can be infinitely many choices of inner products, all of which can support infinitesimal rotations as well as elementary smeared field.

In principle neither of the two routes is unnatural. It is not certain, that continuous symmetries need be realised *exactly* in nature even if observations support their existence to excellent approximation, eg Lorentz symmetry. Symmetries help to exercise tighter control over theoretical frameworks but physical system may not exactly respect the implicit idealization. The in-built, non-invariant dynamics of a polymer quantized system, suggests a particular parametrization of symmetry violation eg the use of the 'trigonometric' operators to build the Hamiltonian. At least in the cases explored, such violations are viable.

The second alternative is anyway needed in the context of theories with first class constraints. It could well be thought of as a multi-step quantization procedure. Just as in a classical theory, specified by an action, the variables we begin with need not represent the physical states (eg when there are constraints). However following a systematic procedure - the Dirac algorithm of constraint analysis - we can arrive at a formulation which is either a theory with a first class constraint algebra or a theory without any constraints. Likewise, one could begin with a set of basic functions on the configuration space forming a Cyl_0, choose an inner product $\langle|\rangle_0$, obtain a Cyl_0^* as well as a Hilbert space H_0 forming a triple: $\mathrm{Cyl}_0 \subset \mathrm{H}_0 \subset \mathrm{Cyl}_0^*$. If the model is satisfactory, we are done. If not, look for a *subspace* $\mathrm{Cyl}_1 \subset \mathrm{Cyl}_0^*$, define a new inner product $\langle|\rangle_1$ and obtain a new triple $\mathrm{Cyl}_1 \subset \mathrm{H}_1 \subset \mathrm{Cyl}_1^*$. Hopefully the process would terminate after a finite number of iterations. This procedure

offers a flexibility to refine the class of observables we wish to be supported on the quantum state space. It is constructive and could help keep the focus on physical observables. This possibility needs to be examined further to see its viability/utility.

We have considered scalar field theory with 'point holonomies' as basic functions generating the commutative C* algebra. Fermions are similar to point holonomies as far as the label sets are concerned. For gauge fields, we will have the $\mathrm{H}_{\mathrm{poly}} := \mathrm{H}_{\mathrm{kin}}$ with the basis labeled by discrete labels. Hence the analogues of $(V, \vec{\lambda})$ will now have embedded graphs and representation labels of the gauge group. One will have to impart a 'manifold structure' for these spaces of labels to attempt a definition of infinitesimal generators in the manner discussed above.

Finally, let us note that these constructions can be carried out in any dimension, for any group - in particular the Lorentz or Poincare groups - with the same qualitative conclusions: unitary, discontinuous representation of the group, non-existence of infinitesimal generators, non-existence of an invariant dynamics which is both non-trivial and physically acceptable and the possibility of subspaces of Cyl^* as possible arenas for further search of acceptable models.

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